International Journal of Mathematics and Computer Applications Research (IJMCAR)
ISSN(P): 2249-6955; ISSN(E): 2249-8060
Vol. 4, Issue 2, Apr 2014, 103- 108

TRANS STELLAR • Journal Publications • Research Consultancy

COMPUTATION OF EIGENVALUES OF A THREE-PARAMETER STURM-LIOUVILLE PROBLEM USING MOMENT METHOD

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ABSTRACT

This paper deals with the numerical solution of eigenvalues of a three-parameter Sturm-Liouville problem using Moment method. The resulting algebraic equations obtained in applying Moment method on the problem are solved to find the rough estimates of the eigenvalues of the problem. Rough estimates are used as starting approximations in the corresponding shooting method to obtain their actual values.

AMS MSC 2010 Code: 34B08, 34B09

KEYWORDS: Eigenvalue, Eigenfunction, Multiparameter, Weighted Residual Method, Moment Method, Shooting Method

1. INTRODUCTION

Multiparameter eigenvalue problems (MPEVP's) are generalisation of one-parameter eigenvalue problems. Separation of variables for a partial differential equation leads to a multiparameter systems of ordinary differential equatioins (Roach(1974)).

It has been observed that one-parameter problem have gained much development both theoretically and numerically in comparison to multiparameter problem. Unlike theoretical aspects (Atkinson (1968), Collatz (1968), Fox *et al.* (1972), Roach (1974), Sleeman (1972)), the numerical methods (Baruah(1987), Baruah (2011)) developed for solving multiparameter problems are very limited. In this paper, we will investigate a three-parameter eigenvalue problem in the form of associated with four point boundary conditions.

$$y''(x) + \{\lambda + \mu f(x) + v g(x)\} y(x) = 0 \tag{1}$$

$$y(a) = y(b) = y(c) = y(d) = 0 \tag{2}$$

Where $a < b < c < d$, $f(x)$ and $g(x)$ are given real valued continuous functions of the independent variable $x \in [a, d]$. The values of the parameters λ, μ and v are called eigenvalues and the non-trivial solutions $y(x)$ are called eigenfunctions of (1)-(2) corresponding to the eigenvalues.

The parameters are to be so chosen that the above equation should have non-trivial solution satisfying boundary conditions. and also the parameters have prescribed number of zeros in each of the intervals (a, b), (b, c) and (c, d) (Sleeman (1972)). The points at which $y(x)$ vanishes in $[a, b]$, $[b, c]$ and $[c, d]$ are called zeros of the eigenfunction. In this paper rough estimates of the eigenvalues are obtained by an application of Moment method which is based on the principle of Weighted-Residual methods. Using the rough estimates as starting approximations in the corresponding shooting method of the problem, actual values are obtained.

2. FORMULATION OF THE METHOD

Initially the complex partial differential equations were solved by finite difference method. But it is difficult to use finite difference method for irregular geometries. To overcome this difficulty a better method was developed which is known as finite element element method. The concept of FEM is based on the classical variational method. But there are some limitations on classical theory to apply on more types of problem while the Weighted-Residual methods provides a strong basis for wider range of problems.

In FEM value of some kind of integral is minimised over the entire domain and the process of minimisation is carried out element wise to get an over all minimum (Baruah(2011)). Moment method which belongs to Weighted Residual methods is also one such approach in which we seek some trial function to obtain an approximate solution. Therefore, we will give a brief account Weighted Residual method followed by Moment method of our present discussion.

3. WEIGHTED RESIDUAL METHOD

The principal idea of weighted residual methods is to find an approximate solution to differential equation in the form (M. K. Jain (1983) and K. S. Rao (2007)),

$$L(u) = f \tag{3}$$

Subject to the boundary conditions such as

$$B_j u = g_j \tag{4}$$

In order to find an approximate solution to the above boundary value problem, we introduce a set of functions of the form

$$\hat{u} = u_0 + \sum_{j=1}^{n} C_j \phi_j \tag{5}$$

where, ϕ_j are the linearly independent functions. Also u_0 satisfy inhomogeneous boundary conditions and ϕ_j satisfy homogeneous boundary conditions. The coefficients C_j are unknowns which are to be determined by solving a system of equations. Since (5) is an approximate function, we observed that, when substituted in (3), it will not satisfy exactly. Thus, we get an error or equation residual denoted by $E(x, a)$ and can be written as,

$$L(\hat{u}) - f = E(x, a) \tag{6}$$

This gives the measure of the extent to which the function $E(x, a)$ satisfies the differential equation. As the number n of the functions $\phi_j(x)$ is increased in successive approximations, we hope that the residue $E(x, a)$ will become smaller. The exact solution is obtained when the residue is identically zero. Since, it is not possible to make the error $E(x, a)$ identically zero, we shall try to make $E(x, a)$ as small as possible in some sense. In the error distributing methods, the error function $E(x, a)$ is made to approximate the zero function, in the sense that each of the weighted integral of the residue of $E(x, a)$ with respect to the weight functions W_j, $j = 1, 2, \dots, n$ is equal to zero.

$$\int E(x, a) W_j \, dx \tag{7}$$

The approximate methods based on (7) are called the weighted residual methods. There are many ways of choosing the weighting functions W_j. Their choices lead to different methods. They are

- Galerkin method
- Least square method
- Method of moments
- Collocation method

As already stated, in this paper we consider Moment method based on this principle as described below.

3.1 Moment Method

In this method, the weighting functions are chosen from family of polynomials such as $w_i = x^i, \quad i = 1, 2, \ldots, n-1$ so that

$$\int E(x,a)x^i dx = 0, \quad i = 1, 2, \ldots, n-1 \tag{8}$$

This gives a system of linear or non-linear equations for the solution of the parameters $a_1, a_2, \ldots, a_n$.

4. NUMERICAL EXAMPLE

The numerical example of the problem (1)-(2) considered here is a three-parameter problem defined by the equation (Collatz (1968)),

$$y''(x) + (\lambda + 2\mu\cos x + 2\nu\cos 2x)y(x) = 0 \tag{9}$$

Subject to the boundary conditions

$$y(0) = y(1) = y(2) = y(3) = 0 \tag{10}$$

Here we choose the approximate solution of (9) satisfying the boundary condition (10) as

$$u(x) = a_1 x(3-x)(x-1)(2-x) + a_2 x^2(3-x)(x-1)(2-x) + a_3 x^3(3-x)(x-1)(2-x) \tag{11}$$

Substituting the approximate solution (11) in the problem (9)-(10), we get,

$$E(x,a) = a_1(12x^2 - 36x + 22) + a_2(20x^3 - 72x^2 + 66x - 12) + a_3(30x^4 - 120x^3 + 132x^2 - 36x) + (\lambda + 2\mu\cos x + 2\nu\cos 2x)\{a_1(x^4 - 6x^3 + 11x^2 - 6x) + a_2(x^5 - 6x^4 + 11x^3 - 6x^2) + a_3(x^6 - 6x^5 + 11x^4 - 6x^3)\} \tag{12}$$

For exemplification the weight functions are considered as $W_1 = 1, W_2 = x$ and $W_3 = x^2$.

Using equation (8) in the intervals $[0,1]$, $[1,2]$ and $[2,3]$ we obtain

$$\int_0^1 E(x,a)\,W_1\,dx = (-0.63333\lambda - 1.11504\mu - 0.72177\nu + 8.00000)a_1 + (-0.28333\lambda - 0.47185\mu - 0.23162\nu + 2.00000)a_2 + (-0.15714\lambda - 0.24983\mu - 0.13516\nu + 2.00000)a_3 = 0 \tag{13}$$

$$\int_1^2 E(x,a)\,W_1\,dx = (0.36666\lambda + 0.05062\mu - 0.65772\nu - 4.00000)a_1 + (0.55000\lambda + 0.04093\mu - 0.99590\nu - 6.00000)a_2 + (0.84286\lambda + 0.01128\mu - 1.53628\nu - 10.00000)a_3 = 0 \tag{14}$$

$$\int_2^3 E(x,a)\,W_1\,dx = (-0.63333\lambda + 1.02845\mu - 0.44310\nu + 8.00000)a_1 + (-1.61667\lambda + 2.65950\mu - 1.23744\nu + 22.00000)a_2 + (-4.15714\lambda + 6.92259\mu - 3.44479\nu + 62.00000)a_3 = 0 \tag{15}$$

$$\int_0^1 E(x,a)\,W_2\,dx = (-0.28333\lambda - 0.47185\mu - 0.23162\nu + 2.00000)a_1 + (-0.15714\lambda - 0.24983\mu - 0.08969\nu + 2.00000)a_2 + (-0.09881\lambda - 0.15116\mu - 0.03757\nu + 2.00000)a_3 = 0 \tag{16}$$

$$\int_1^2 E(x,a)\,W_2\,dx = (0.55000\lambda + 0.04093\mu - 0.99598\nu - 6.00000)a_1 + (0.84286\lambda + 0.01129\mu - 1.53628\nu - 10.00000)a_2 + (1.31786\lambda - 0.05832\mu - 2.41119\nu - 18.00000)a_3 = 0 \quad (17)$$

$$\int_2^3 E(x,a)\,W_2\,dx = (-1.61667\lambda + 2.65950\mu - 1.23744\nu + 22.00000)a_1 + (-4.15714\lambda + 6.92259\mu - 3.44479\nu + 62.00000)a_2 + (-10.76548\lambda + 18.13280\mu - 9.57093\nu + 178.00000)a_3 = 0 \quad (18)$$

$$\int_0^1 E(x,a)\,W_3\,dx = (-0.15714\lambda - 0.24983\mu - 0.08969\nu + 0.73333)a_1 + (-0.09881\lambda - 0.15116\mu - 0.03757\nu + 1.43333)a_2 + (-0.06746\lambda - 0.09993\mu - 0.01556\nu + 1.68571)a_3 = 0 \quad (19)$$

$$\int_1^2 E(x,a)\,W_3\,dx = (0.84286\lambda + 0.01129\mu - 1.53628\nu - 9.26667)a_1 + (1.31786\lambda - 0.05832\mu - 2.41119\nu - 16.90000)a_2 + (2.09921\lambda - 0.20573\mu - 3.84614\nu - 32.31429)a_3 = 0 \quad (20)$$

$$\int_2^3 E(x,a)\,W_3\,dx = (-4.15714\lambda + 6.92259\mu - 3.44480\nu - 60.733333)a_1 + (-10.76548\lambda + 18.13280\mu - 9.57093\nu + 174.76667)a_2 + (-28.06746\lambda + 47.78134\mu - 26.56431\nu + 509.68570)a_3 = 0 \quad (21)$$

Eliminating the parameters a_1, a_2 and a_3 from the equations (13)-(21) taking three at a time we get different algebraic equations of fourth order.The algebraic equations are solved (S. S. Sastry (2006)) to get the starting values of the eigenvalues (λ, μ, ν) of the problem (9)-(10) as shown in the table-1 below.

Table 1: Starting Appox for Shooting Method

Equations Considered to Eliminate a_1, a_2 and a_3	Approximate Solutions of (λ, μ, ν)
(13), (14), (15) (16), (17), (18) (19), (20), (21)	(0.39355, (-0.17528, 7.66561)
(13), (20), (21) (13), (15), (17) (14), (15), (16)	(0.83488, -0.32253, 3.57482)
(14), (15), (16) (15), (16), (17) (16), (17), (18)	(0.06837, 0.05869, 2.43435)
(13), (14), (15) (13), (16), (17) (13), (14), (16)	(2.93326, -0.08234, 19.90332)
(13), (18), (19) (15), (17), (21) (14), (16), (20)	(-1.58854, -0.22714, 4.17804)
(13), (16), (21) (14), (17), (19) (15), (18), (20)	(0.22315, 0.02355, 8.76451)
(13), (17), (21) (15), (17), (19) (15), (16), (18)	(1.65694, -0.01337, -2.05649)
(16), (20), (21) (15), (19), (20) (14), (18), (19)	(1.63134, 1.09124, -4.53315)
(14), (15), (19) (18), (19), (20) (13), (16), (21)	(-0.68440, -0.56030, 16.81688)
(13), (17), (21) (16), (19), (21) (14), (17),(21)	(0.31307, -0.00003, 8.58012)
(13), (17), (18) (16), (20), (21) (15), (17), (19)	(1.03074, -1.66158, 4.55446)
(13), (16), (19) (14), (17), (20) (15), (17), (21)	(0.10510, 5.64528, -12.95556)

The rough estimates from **Table 1** are used as starting approximation in the corresponding shooting metrhod to obtain the actual values of the eigenvalues (λ, μ, ν) of the problem (1)-(2). The distinct actual eigenvalues are found as (9.85032, 1.17066, 0.21466), (20.32325, -7.60925, 4.53812), (37.99081, -22.74049, 12.21601), (49.93017, -1.15909, -29.31049), (59.66886, -14.81822, -21.84090), (84.06648, -34.78913, -11.11194), the corresponding eigenfunctions have number of zeros in the intervals (0, 1), (1, 2) and (2, 3) are (0, 0, 0), (0, 0, 1), (0, 0, 3), (1, 3, 1), (1, 3, 3), (1, 3, 5) respectively.

In the numerical example illustrated above the starting approximations of the eigenvalues depend on the choice of approximate solution. The problem defined in the domain $[a, d]$ are considered separately in the intervals $[a, b]$, $[b, c]$ and $[c, d]$ so that enough algebraic equations can be constructed to obtain the unknowns a_j's together with λ, μ and ν.

4. CONCLUSIONS

Compared to theoretical aspects, numerical treatment of three-parameter eigenvalue problems are still very few and hence some contribution in this area are always in need. In this paper eigenvalues of a three-parameter Sturm-Liouville problem is computed using Moment method, which fall under Weighted-Residual method. One of the most important advantage of this method is that it can be applied to both non-selfadjoint and non-linear problems(Finalayson (1966)).

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